Part 1. Fundamentals — The Spherical Wave Functions

Expressing Acoustic Radiation with Spherical Wave Functions

At a given frequency, the acoustic pressure field generated by a loudspeaker can be expressed as a summation of a set of *basis functions* multiplied by their weighting factors. For our problem at hand, a very suitable choice for these basis functions is the spherical wave functions in the spherical coordinate system.

The radiation pattern of the sound pressure generated by a sound source can be expressed in two related forms [1, p.20][1]:

$$\hat{p}(r,\theta,\phi;\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A_{mn} j_n(kr) + B_{mn} y_n(kr)\right] Y_n^m(\theta,\phi) \tag{1.1}$$

and

$$\hat{p}(r,\theta,\phi;\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A_{mn} h_n^{(1)}(kr) + B_{mn} h_n^{(2)}(kr)\right] Y_n^m(\theta,\phi) \tag{1.2}$$

where	$\hat{p}(r,\theta,\phi;\omega)$	is the complex acoustic pressure amplitudes at coordinates r,θ,ϕ and at angular frequency ω;
	r,θ,ϕ	are the spherical coordinates;
	ω	is the angular frequency, with $\omega = 2\pi f$;
	$j_n(kr)$, $y_n(kr)$	are the spherical Bessel functions of the first and second kinds;
	$h_n^{(1)}(kr)$, $h_n^{(2)}(kr)$	are the spherical Hankel functions of the first and second kinds;
	k	is the acoustic wavenumber and is defined as $k = \omega/c$ with $c =$ speed of sound;
	$Y_n^m(\theta,\phi)$	is the spherical harmonics;
	m, n	are the degree and order of the spherical harmonics; and
	A_{mn}, B_{mn}	are the weighting coefficients

[1]The reader needs not be scared by the exotic sounding names of the functions in these equations. These functions, or their major building blocks, are available in many numerical libraries. For Python, the spherical Bessel functions and the spherical Harmonics are available as `scipy.special.spherical_jn()`, `scipy.special.spherical_yn()` and `scipy.special.sph_harm()`. The spherical Hankel functions can be calculated using equations 1.3 and 1.4.
For the adventurous, there is also the GNU Scientific Library. Please consult the relevant documentation pages and the NIST Digital Library of Mathematical Functions (DLMF).

The relationships between the spherical Bessels and the spherical Hankel functions are:

$$h_n^{(1)}(kr) = j_n(kr) + iy_n(kr) \tag{1.3}$$

$$h_n^{(2)}(kr) = j_n(kr) - iy_n(kr) \tag{1.4}$$

where $i = \sqrt{-1}$

The spherical Bessel and spherical Hankel functions describe the acoustic pressure field dependency on the radial coordinate (i.e. distance), and the spherical harmonics describe its dependency on the two angular coordinates (directions).

Equation (1.1) expresses the acoustic pressure wave as standing waves in the interior region; whereas equation (1.2) expresses it as traveling waves in the exterior region [1, p.20]. For the exterior region, the spherical Hankel functions of the first kind $h_n^{(1)}(kr)$ describes the outgoing wave and the spherical Hankel functions of the second kind $h_n^{(2)}(kr)$ describes the incoming waves.

The ability to separate the acoustics pressure wave as standing waves in the interior region and as traveling waves in the exterior region allows us to separate the acoustic sources interior to a *source free* measurement zone from those that are outside. Sound field separation is covered in Part 3.

Note on the Convention of the Time and Frequency Domains Relationship [2][3, p.75]

Note that the convention used in this report is different from that used in Klippel's [4, 5, 6, 7] and a number of other publications. In this report, the form of the time-harmonic function used is $e^{-i\omega t}$, i.e. the frequency domain and time domain are related by $p(t) = \hat{p}(\omega)e^{-i\omega t}$. This is the convention preferred by physicists which include acousticians, whereas electrical engineers usually prefer $e^{i\omega t}$. This is the reason some papers (including those of Klippel's) state that the spherical Hankel function of the second kind $h_n^{(2)}(kr)$ depicts the outgoing wave.

When measuring an actual loudspeaker, the sound pressure is measured by a microphone and the acquired signal is in time-domain. The signal must be Fourier transformed into the frequency domain for processing when using the methods described in this report. The time and frequency domains relationship convention matters here too. Most computer FFT library functions follow the EE's convention. To convert between the two conventions, simply take the complex conjugates of the complex FFT coefficients. When using the FFTW library, choose `FFTW_FORWARD` or `FFTW_BACKWARD` based on the convention.[8] [9].

Note on the Spherical Coordinate System

Again, we have an issue with conventions. Physicists and mathematicians have different conventions for the spherical coordinate system. Physicists refer to the colatitudinal angle as θ and the azimuthal angle as ϕ. Mathematicians do it the other way around, as they prefer to be consistent with the cylindrical coordinate system. The cylindrical coordinate

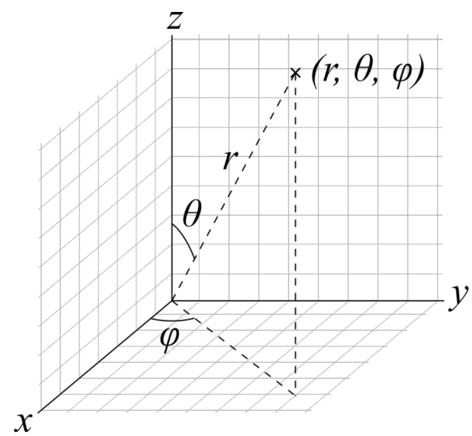

Figure 1: The Spherical Coordinate System Used by Physicists [10]

system has one angular coordinate, and it was assigned the Greek letter θ. It happens that this angular coordinate in the cylindrical coordinate system is the azimuthal angle.

In this report, the physicists' convention is used. Its relationship with the Cartesian coordinate system is shown in Figure 1. This convention is (probably) used in all of the materials referenced in this report. However, when standard computer library functions are used to compute the spherical harmonics, these functions sometimes follow the mathematicians' convention (e.g. the Python Scipy function `scipy.special.sph_harm()`). In these cases, one will need to swap θ and ϕ in the function call.

Spherical Harmonics

The spherical harmonics is written as $Y_n^m(\theta, \phi)$. n is the order, and is an integer with a range of 0 to ∞. m is the degree and its range is $-n \leq m \leq n$. For $n = 0$, there is only one spherical harmonic $Y_0^0(\theta, \phi)$, which represents a monopole source. For $n = 1$, there are three spherical harmonics, $Y_1^{-1}(\theta, \phi)$, $Y_1^0(\theta, \phi)$ and $Y_1^1(\theta, \phi)$, which represent three dipole sources in three mutually perpendicular directions. For $n = 2$, there are 5 spherical harmonics representing 5 quadrupole sources. Figure 2 shows graphically the spherical harmonics from $n = 0$ to 5 (top to bottom).

The spherical harmonics provide the directivity dependence on the angular coordinates. We can express the angular directivity of the sound wave generated by a loudspeaker by a combination of these spherical harmonics. The more complex the directivity pattern, the more of these spherical harmonics (i.e. inclusion of higher and higher orders) is required.

The values returned from the spherical harmonics functions are complex numbers. The color shaded shapes in Figure 2 are generated using the following scheme:

- An angular (plotting) mesh of θ and ϕ is generated. r depend on the values of $Y_n^m(\theta, \phi)$.
- $r = \begin{cases} |\mathrm{Im}\left[Y_n^m(\theta, \phi)\right]| & m < 0 \\ |\mathrm{Re}\left[Y_n^m(\theta, \phi)\right]| & m \geq 0 \end{cases}$

- Color values $= \begin{cases} \mathrm{Im}\left[Y_n^m(\theta,\phi)\right] & m < 0 \\ \mathrm{Re}\left[Y_n^m(\theta,\phi)\right] & m \geq 0 \end{cases}$

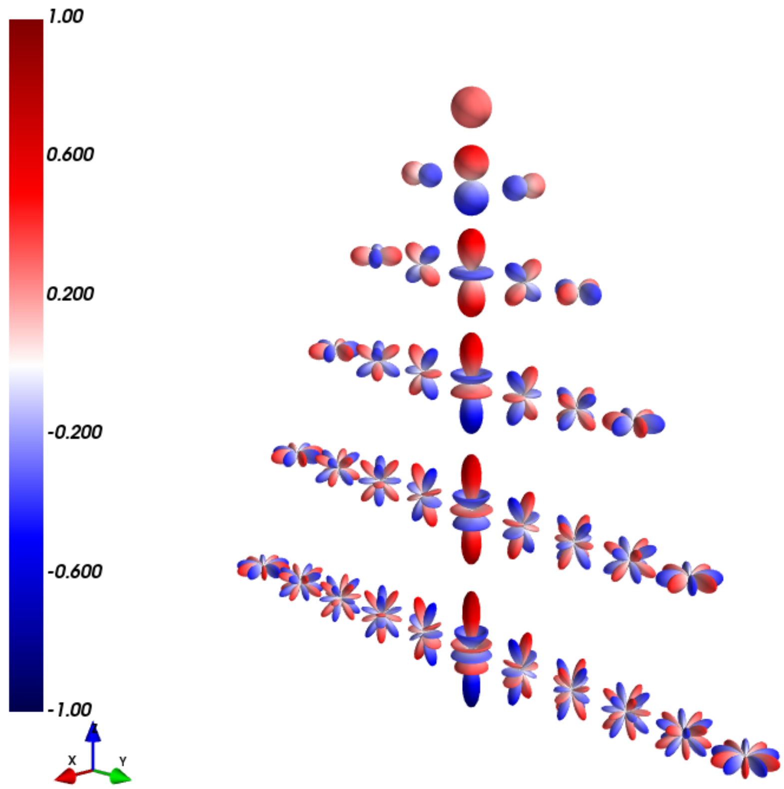

Figure 2: Spherical Harmonics, $n = 0$ to 5

You can find similar looking plots of the spherical harmonics in Klippel's papers [4, fig. 2], [7, slide 25] and in Wikipedia [11]. May be we are in the right direction!

References

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[8] FFTW FAQ, Section 3, Question 3.7

[9] FFTW Documentation, Section 4.8.1, The 1d Discrete Fourier Transform (DFT)

[10] By Andeggs - Own work, Public Domain

[11] Wikipedia page on Spherical Harmonics